Here the exact expression for the integrals of a power of r times the wheight function is developed. Let first b and c denote the sides of the rectangle, so normalized that $\sqrt{b^2+c^2}=1$. By integration by parts and a change of variable it is found that

$$\mathrm{I}_n = \int_0^1 r^n p(r) = \frac{c^{n+2}}{n+2}\int \cosh^{n+1} t \,\mathrm{d}t + \frac{b^{n+2}}{n+2}\int \cosh^{n+1} t \,\mathrm{d}t$$

The lower limits in the integrals at the right hand side of the equation are both zero and the upper ones the values y such that $\cosh y = \frac{1}{c}$ and $\frac{1}{b}$ respectively. These values are $\log\frac{1+b}{c}$ and $\log\frac{1+c}{b}$, and the values of $\sinh y$ are $\frac{b}{c}$ and $\frac{c}{b}$.

For $n=0$ the result is

$$\int p = \int ds = \mathrm{S} = bc.$$

The integral of $\cosh^{n+1}$ can be found by developping the power $(e^t+e^{-t})^{n+1}$, which avoids integration by parts, or by iterated integration of $\cosh^{n+1}$, $\cosh^{n-1}$, $\cosh^{n-3}$, etc. The expressions arising are different for even and odd n.

By taking the later approach it is found, for n even,

$$\begin{aligned}\frac{(n+1)(n+2)\mathrm{I}_n}{bc} = 2 + \frac{n}{n-1}(b^2+c^2) + \frac{n(n-2)}{(n-1)(n-3)}(b^4+c^4) + \cdots \\ + \frac{n(n-2)\cdots 4\cdot 2}{(n-1)(n-3)\cdots 3\cdot 1}(b^n+c^n)\end{aligned} \tag{1}$$

Taking into account that $b^2+c^2=1$ we have

$$b^n + c^n = b^{n-2} + c^{n-2} - (bc)^2(b^{n-4}+c^{n-4}). \tag{2}$$

Applying this recursively from the last term of (1) backwards that expression is reduced to

$$\begin{aligned}\frac{(n+1)(n+2)\mathrm{I}_n}{bc} = (n+2) - (bc)^2\,\frac{n(n-2)}{(n-1)}\Bigg\{\frac{(n-4)\cdots 2}{(n-3)\cdots 3}(b^{n-4}+c^{n-4}) \\ + \frac{(n-4)\cdots 4}{(n-3)\cdots 5}(b^{n-6}+c^{n-6}) + \cdots + \frac{(n-4)}{(n-3)}(b^2+c^2) + 2\Bigg\}\end{aligned}$$

Applying (2) again:

$$\begin{aligned}\frac{(n+1)(n+2)\mathrm{I}_n}{bc} = (n+2) - (bc)^2\,\frac{n(n-2)}{(n-1)}\Bigg\{\frac{(n+2)}{3} \\ - \frac{(bc)^2}{3}\,\frac{(n-4)(n-6)}{n-3}\Bigg\{\frac{(n-8)\cdots 2}{(n-5)\cdots 5}(b^{n-8}+c^{n-8}) + \frac{(n-8)\cdots 4}{(n-5)\cdots 7}(b^{n-10}+c^{n-10}) \\ + \cdots + \frac{(n-8)}{(n-5)}(b^2+c^2) + 2\Bigg\}\Bigg\}\end{aligned}$$

or which is the same

$$\frac{(n+1)\mathrm{I}_n}{bc} = 1 - \frac{n(n-2)}{3(n-1)}\,(bc)^2 + \frac{1}{n+2}\,\frac{n(n-2)(n-4)(n-6)}{3(n-1)(n-3)}\,(bc)^4 \left\{ \frac{(n-8)\cdots 2}{(n-5)\cdots 5}\,(b^{n-8}+c^{n-8}) + \cdots + \frac{(n-8)}{(n-5)}\,(b^2+c^2) + 2 \right\}$$

This process may be continued as it can easily be seen. The series of fractions within the brakets will be of the form

$$a_1 = \frac{2\cdot 4\cdot 6\cdots}{r(r+2)(r+4)\cdots}\,, \qquad a_2 = \frac{4\cdot 6\cdots}{(r+2)(r+4)\cdots}\,, \qquad \text{etc.}$$

When the sums $a_1 + a_2$, $a_1 + a_2 + a_3$, etc. are taken the result of each of them is the next fraction in the series with the smallest factor of its denominator replaced by r. The sum of all the fractions will therefore be

$$\frac{(n-(2r-2))}{r} + 2 = \frac{n+2}{r}\,.$$

The $(n+2)$ term cancels out with the one outside the brackets. The remaining partial sumations, which multiply $(bc)^2$, will all include the factors $(n-(2r-2))(n-2r)/(n-r)r$, which together with $(bc)^2$ itself are extracted out of the brackets, and the process continues.

The final formula for the integral is

$$\frac{\mathrm{I}_n}{bc} = \frac{1}{n+1} - \frac{n(n-2)}{3(n+1)(n-1)}\,(bc)^2 + \frac{n(n-2)(n-4)(n-6)}{5¡(n+1)(n-1)(n-3)}\,(bc)^4 - \frac{n(n-2)(n-4)(n-6)(n-8)(n-10)}{7¡(n+1)(n-1)(n-3)(n-5)}\,(bc)^6 + \cdots \tag{3}$$

where $r¡$ means $1{\cdot}3{\cdot}5\cdots r$.

For odd values of n let $\mathrm{C} = \frac{1}{b}\log\frac{1+b}{c}$ and $\mathrm{B} = \frac{1}{c}\log\frac{1+c}{b}$. The integration yields

$$\frac{(n+1)(n+2)\mathrm{I}_n}{bc} = 2 + \frac{n}{n-1}\,(b^2+c^2) + \frac{n(n-2)}{(n-1)(n-3)}\,(b^4+c^4) + \cdots + \frac{n(n-2)\cdots 3}{(n-1)(n-3)\cdots 2}\,(b^{n-1}+c^{n-1}) + \frac{n(n-2)\cdots 3}{(n-1)(n-3)\cdots 2}\,(c^{n+1}\mathrm{C} + b^{n+1}\mathrm{B}) \tag{1'}$$

which cannot be simplified like the formula for even n.

The first values of $\int r^n p(r)/bc$ are

n	$\mathrm{I}_n\,/\,bc$	n	$\mathrm{I}_n\,/\,bc$
2	$1/3$	1	$1/3 + 1/6(c^2\mathrm{C} + b^2\mathrm{B})$
4	$1/5 - 8/45(bc)^2$	3	$7/40 + 3/40(c^4\mathrm{C} + b^4\mathrm{B})$
6	$1/7 - 8/35(bc)^2$	5	$1/336\big(41 - 30(bc)^2 + 15(c^6\mathrm{C} + b^6\mathrm{B})\big)$

Instead of computing the coefficients for different ratios b/c and taking an average it is easier and not worse to take the coefficients for a particular ratio. I have chosen to take the ratio $8/7$ for the p polinomials and $3/2$ for the q polinomials, for these are the simple ratios that yield the values of the coefficients of the second polinomials $3, -2$ and $4, -3$ respectively, up to two decimal places (and in the first case up to three indeed). These are the definitive polinomials. To the right of each polinomial the quadratic mean value of it is shown, computed for a photograph with the side ratio $4/3$ (but, as was noted, the values vary little for other ratios):

$$\begin{array}{rr}
r & 0.58 \\
3r^2 - 2r & 0.28 \\
9r^3 - 11.4r^2 + 3.4r & 0.17 \\
29.2r^4 - 53.1r^3 + 30.1r^2 - 5.2r & 0.13 \\
95.8r^5 - 225.4r^4 + 187.1r^3 - 63.9r^2 + 7.4r & 0.096 \\
320.3r^6 - 922.1r^5 + 1004.9r^4 - 511.4r^3 + 119.2r^2 - 9.9r & 0.077
\end{array}$$

$$\begin{array}{rr}
r^2 & 0.40 \\
4r^3 - 3r^2 & 0.22 \\
14.5r^4 - 20.3r^3 + 6.8r^2 & 0.14 \\
53.5r^5 - 107.8r^4 + 69.5r^3 - 14.2r^2 & 0.10 \\
197.5r^6 - 511.4r^5 + 476.9r^4 - 188.2r^3 + 26.2r^2 & 0.082
\end{array}$$

And here are the polynomials for weighted odd components, computed for a ratio b/c equal to $\sqrt{3}/1$, for which the coefficients of the second polynomial of the series p are exactly $2, -1$ and those of the second polynomial of the series q are $2.5, -1.5$. To the right of each polinomial the quadratic mean value of it is shown, computed for a photograph with the side ratio $\sqrt{3}/1$:

$$\begin{array}{rr}
r & 0.58 \\
2r^3 - r & 0.26 \\
4.8r^5 - 4.7r^3 + 0.9r & 0.14 \\
12.8r^7 - 19.1r^5 + 8.2r^3 - 0.9r & 0.093 \\
38.4r^9 - 76.2r^7 + 50.5r^5 - 12.6r^3 + 0.9r & 0.069 \\
119.5r^{11} - 296.7r^9 + 268r^7 - 106.5r^5 + 17.6r^3 - 0.9r & 0.055
\end{array}$$

$$\begin{array}{rr}
r^2 & 0.41 \\
2.5r^4 - 1.5r^2 & 0.20 \\
6.4r^6 - 7.2r^4 + 1.8r^2 & 0.11 \\
19.1r^8 - 31.6r^6 + 15.7r^4 - 2.2s^2 & 0.080 \\
60.4r^{10} - 131r^8 + 97.8r^6 - 28.9r^4 + 2.7r^2 & 0.061
\end{array}$$

The expressions for I_n/bc when n is even are rational. More exactly, if $(bc)^2$ is a rational number the value of I_n/bc will be a rational number too. Therefore, the coefficients of the polynomials are given by linear equations in which the coefficients are rationals, and are therefore themselves rational. But the expressions get very complicated soon. For a ratio b/c equal to $\sqrt{3}/1$, the value of $(bc)^2$ is 3/16 and the values of the integrals are:

$n:$	0	2	4	6	8	10	12	14	16	18	20
$\mathrm{I}_n/bc:$	1	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{10}$	$\frac{209}{3150}$	$\frac{65}{1386}$	$\frac{971}{28028}$	$\frac{227}{8580}$	$\frac{89}{4290}$	$\frac{2021}{121550}$	$\frac{579347}{42678636}$

The exact polynomials p_1, p_2, p_3, q_1, q_2 and q_3 would be

$$r, \qquad 2r^3 - 1r, \qquad \tfrac{105}{22}r^5 - \tfrac{103}{22}r^3 + \tfrac{10}{11}r, \qquad\qquad r^2, \qquad \tfrac{5}{2}r^4 - \tfrac{3}{2}r^2, \qquad \tfrac{28875}{4486}r^6 - \tfrac{32235}{4486}r^4 + \tfrac{7846}{4486}r^2$$

but for p_4 and q_4 the denominators are already 36 590 794 and 3 458 217 302.